

Rolling in the Higgs Model and Elliptic Functions

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ABSTRACT: Asymptotic methods in nonlinear dynamics are used usually to improve perturbation theory results in the oscillations regime. However, for some problems of nonlinear dynamics, particularly in the case of Higgs (Duffing) equation and the Friedmann cosmological equations, not only small oscillations regime is of interest but also the regime of rolling (climbing), more precisely the rolling from a top (climbing to the top). In the Friedman cosmology, where the slow rolling regime is often used, the rolling from a top (not necessary slow) is of interest too.

In the present work a method for approximate solution to the Higgs equation in the rolling regime is presented. It is shown that in order to improve perturbation theory in the rolling regime it turns out to be effective not to use an expansion in trigonometric functions as it is done in case of small oscillations but use expansions in hyperbolic functions instead. This regime is investigated using the representation of the solution in terms of elliptic functions. An accuracy of the corresponding approximation is estimated.

KEYWORDS: oscillations, rolling regime, Higgs model.

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1. Introduction

Known asymptotic methods in nonlinear dynamics such as the Krylov-Bogoliubov averaging method, the Lyapunov method of slow varying parameters, the Poincare and the Van der Pol methods, and KAM-theory are used to improve perturbation theory results when describing small oscillations, see [1, 2, 3, 4]. Even the term itself - nonlinear mechanics - is frequently considered as a synonym to nonlinear oscillations theory, see [1].

However, in case of the Higgs equation in field theory [5], the Friedman equation in cosmology [6, 7, 8] and some other problems of nonlinear dynamics not only regime of small oscillations is of interest but also the rolling regime, see for example [9, 10] and references within.

Small perturbations of quasi-periodical motions usually are considered as perturbations to the linear equation of the harmonic oscillator:

$$\ddot{q} + \omega^2 q = 0, \quad (1.1)$$

where $q = q(t)$ is a real-valued function of time t and $\omega > 0$.

Rolling (or climbing) regime by definition is a small perturbation of the following linear equation:

$$\ddot{q} - \mu^2 q = 0, \quad (1.2)$$

where $\mu > 0$.

In the present work a method to study nonlinear systems in the rolling regime is presented. This method can be considered as an hyperbolic analogue of the averaging method [2]. We consider it in an example of the Higgs equation

$$\ddot{q}(t) - \mu^2 q(t) = -\epsilon q^3(t), \quad \mu > 0, \quad \epsilon > 0, \quad (1.3)$$

where ϵ is small coupling constant. The method gives approximate solutions using expansions of the exact solution in hyperbolic functions. The following approximate solution is found in Theorem 1 in Sect. 7.3:

$$q_{appr}(t) = A \sinh\left(t\mu\left(1 + \frac{3}{8} \frac{\epsilon A^2}{\mu^2}\right)\right) - \frac{1}{32} \frac{\epsilon^2 A^3}{\mu^4} \sinh(3t\mu), \quad (1.4)$$

where A is an arbitrary constant. One has the bound

$$|q(t) - q_{appr}(t)| \leq C\epsilon^{2-\sigma}, \quad 0 < \sigma < 3/4 \quad (1.5)$$

which is valid for time in the interval

$$0 \leq t \leq c_1 \log \frac{c_2}{\epsilon} \quad (1.6)$$

that expanding with the vanishing ϵ .

It is well-known that simple equations of nonlinear dynamics allow exact solutions in terms of elliptic functions. Correspondence between these exact solutions and asymptotic averaging method is considered in the book [2], where an equation for anharmonic oscillator with real frequency is investigated

$$\ddot{q}(t) + \omega^2 q(t) = -\epsilon q^3(t), \quad \omega > 0, \quad \epsilon > 0. \quad (1.7)$$

In [2] numerical comparison of approximated and exact solutions is presented.

In the present work, unlike [2], the equation (1.3) with the imaginary frequency is considered. It also has a representation in terms of elliptic functions. The rolling regime for solutions to this equations is considered. Equations (1.7) and (1.3) are connected via change $\omega = i\mu$. However, the averaging method is not applicable to study the rolling regime for solutions to (1.3). We use another method that consists in expansion of solutions in hyperbolic functions.¹ Applications of approximate solutions in functional mechanics are considered in [11, 12, 13].

It is well known that by a shift at a constant $q(t) = x(t) + \mu/\sqrt{\epsilon}$, the equation (1.3) is converted into the equation describing nonlinear oscillations

$$\ddot{x} + 2\mu^2 x = -\epsilon x^3 - 3\sqrt{\epsilon}\mu x^2. \quad (1.8)$$

However this transformation does not allow to investigate the rolling mode more effectively then explicit consideration of the initial Higgs equation.

The paper is organized as follows. In Sect.2 we discuss a hyperbolic analogue of the averaging method. In Sect.3 solutions to the Higgs equation in terms of elliptic functions is presented. Expansions of elliptic functions in hyperbolic ones are presented in Sect. 4 where bounds on the n-mode approximations are also presented. A rearrangement of the expansion of the elliptic function in the form suitable to make connection with the series obtained via the hyperbolic analogue of the averaging method is exhibited in Sect.5. The identity of these two series is noted in Sect.5. Estimations of precision of hyperbolic approximations are presented in Sect.6. In Appendix some details of the proofs of Lemma 1 and Theorem 2 are presented.

The paper is based on [17, 18].

¹It would be interesting to consider possible correspondence of the method of expansion in hyperbolic functions used in the present work with small t with the first Lyapunov method development [4] that is effective with $t \rightarrow \infty$.

2. Hyperbolic analogue of the averaging method

The simplest equation considered in nonlinear dynamics has the following form:

$$\ddot{q} + \omega^2 q = \epsilon f(q, \dot{q}), \quad (2.1)$$

where $q = q(t)$ is a real-valued function of time t , frequency ω is a positive number, ϵ is a small parameter.

In applications the following equation is also considered

$$\ddot{q} - \mu^2 q = \epsilon f(q, \dot{q}), \quad (2.2)$$

where $\mu^2 > 0$. It comes from (2.1) via a change $\omega = i\mu$. This equation is called an equation of rolling. For instance, the Higgs equation has this form (see [5]) and also the Friedman equation with a tachyon field in cosmology (see [10]).

By analogy with the Krylov-Bogoliubov expansion method, used to solve equations describing oscillations (2.1), we will look for general solution to the equation describing rolling (2.2) in the form of expansion

$$q = a \sinh \psi + \epsilon u_1(a, \psi) + \epsilon^2 u_2(a, \psi) + \dots \quad (2.3)$$

Here a, ψ as functions of time that are determined by differential equations

$$\dot{a} = \epsilon A(a, \epsilon) = \epsilon A_1(a) + \epsilon^2 A_2(a) + \dots, \dot{\psi} = \mu + \epsilon B(a, \epsilon) = \mu + \epsilon B_1(a) + \epsilon^2 B_2(a) + \dots \quad (2.4)$$

Let us consider equation

$$\ddot{q}(t) - \mu^2 q(t) = -\epsilon q^3(t), \quad \mu > 0, \quad \epsilon > 0. \quad (2.5)$$

We look to solution of the form

$$q(t) = \mathfrak{A} \sinh(t\mu\mathfrak{m}) + \dots, \quad (2.6)$$

and define parameters $\mathfrak{A}, \mathfrak{m}$ in a way to keep the functional form of the solution after substitution in the equation (2.5). We get on the right-hand side of (2.5):

$$-\epsilon (\mathfrak{A} \sinh(t\mu\mathfrak{m}) + \dots)^3 = -\epsilon \left(\frac{1}{4} \mathfrak{A}^3 \sinh(3t\mu\mathfrak{m}) + 3 \sinh(t\mu\mathfrak{m}) + \dots \right) \quad (2.7)$$

On the left-hand side of (2.5) we obtain:

$$(\partial^2 - \mu^2)(\mathfrak{A} \sinh(t\mu\mathfrak{m}) + \dots) = \mathfrak{A}\mu(\mathfrak{m}^2 - 1) \sinh(t\mu\mathfrak{m}) + \dots \quad (2.8)$$

We see that in order to get an equality we have to equate

$$\mathfrak{A}\mu^2(1 - \mathfrak{m}^2) = -\epsilon \frac{3}{4} \mathfrak{A}^3, \quad (2.9)$$

Therefore it follows that

$$\mathbf{m}^2 = 1 + \frac{3}{4} \frac{\epsilon \mathfrak{A}^2}{\mu^2}. \quad (2.10)$$

In order to keep the form of the approximation in the right-hand side and the left-hand sides of (2.5) one has to take the next term of the approximation in the form

$$q(t) = \mathfrak{A} \sinh \left(t\mu \left(1 + \frac{3}{8} \frac{\epsilon \mathfrak{A}^2}{\mu^2} \right) \right) + \mathfrak{A}_1 \sinh(3t\mu) + \dots \quad (2.11)$$

Taking into account (2.7) we get

$$\mathfrak{A}_1 = -\frac{1}{32} \frac{\epsilon^2 \mathfrak{A}^3}{\mu^4}. \quad (2.12)$$

Approximation (2.11),(2.12) is considered in [9].

The described procedure can be carried on and it turns out that for equation (2.5) the procedure leads to the representation of the solution with initial data $\dot{q}(0) = 0$ in the form of an infinite sum of n -mode terms

$$q(t) = \sum_{n=0}^{\infty} \mathfrak{s}_n(t), \quad (2.13)$$

where

$$\mathfrak{s}_0(t) = \mathfrak{A} \sinh(\mu \mathbf{m} t), \quad \mathbf{m} = \mathbf{m}(\lambda), \quad (2.14)$$

$$\mathfrak{s}_n(t) = \mathfrak{A} \mathfrak{a}_n(\lambda) \sinh((2n+1)\mu \mathbf{m} t), \quad n = 1, \dots \quad (2.15)$$

Here $\mathbf{m}(\lambda)$ and $\mathfrak{a}_n(\lambda)$ are power series of dimensionless parameter

$$\lambda = \frac{\mathfrak{A}^2 \epsilon}{\mu^2}, \quad (2.16)$$

$$\mathbf{m}(\lambda) = 1 + \mathbf{m}_1 \lambda + \dots + \mathbf{m}_n \lambda^n + \dots \quad (2.17)$$

$$\mathfrak{a}_n(\lambda) = \mathfrak{a}_{n0} \lambda^n + \mathfrak{a}_{n1} \lambda^{n+1} + \dots + \mathfrak{a}_{nn} \lambda^{2n} + \dots \quad (2.18)$$

The solution depends on parameters ϵ, μ , and it is specified by the only parameter \mathfrak{A} . Within this notation the first terms of (2.13) are written in the form

$$\begin{aligned} q(t) = & \mathfrak{A} \sinh \left(\mu \left(1 + \frac{3\mathfrak{A}^2}{8\mu^2} \epsilon - \frac{15\mathfrak{A}^4}{2^8\mu^4} \epsilon^2 \right) t \right) \\ & - \mathfrak{A} \left(\frac{\epsilon \mathfrak{A}^2}{32\mu^2} - \frac{21\epsilon^2 \mathfrak{A}^4}{1024\mu^4} \right) \sinh \left(3\mu \left(1 + \frac{3\mathfrak{A}^2}{8\mu^2} \epsilon \right) t \right) + \frac{21\epsilon^2 \mathfrak{A}^5}{1024\mu^4} \sinh(5\mu t) + \dots \end{aligned} \quad (2.19)$$

This series can be formally obtained for the result of the averaging method (see (3.7) below) by the change $\omega \rightarrow i\mu$, $A \rightarrow -i\mathfrak{A}$.

For the Higgs equation with friction:

$$(\partial^2 + 2\epsilon h \partial - \omega^2)q + \epsilon q^3 = 0, \quad q(0) = 0 \quad (2.20)$$

an approximate solution has the form

$$q(t) = \mathfrak{A} e^{-\epsilon h t} \sinh \left(\mu \left(t - \frac{3\mathfrak{A}^2}{16\mu^2 h} (1 - e^{-2\epsilon h t}) \right) \right) \quad (2.21)$$

$$- \epsilon \frac{\mathfrak{A}^3}{32\mu^2} e^{-3\epsilon h t} \sinh \left(3\mu \left(t - \frac{3\mathfrak{A}^2 (1 - e^{-2\epsilon h t})}{16\mu^2 h} \right) \right) \quad (2.22)$$

The method developed in this work can be also used to investigate a rolling mode (not necessarily slow one) for the Friedman equation either in local or in nonlocal cosmology, see [8, 9]:

$$\ddot{q} + \epsilon_1 \sqrt{\frac{\dot{q}^2}{2} \pm \omega^2 \frac{q^2}{2}} + \epsilon_2 V(q) \pm \omega^2 q + \epsilon_2 V'(q) = 0. \quad (2.23)$$

3. Higgs equation and elliptic functions

3.1 Solution to the Duffing equation

In this section the known solutions to some nonlinear equations are presented.

As it is known the equation of the form

$$\ddot{q} + V'(q) = 0$$

has the integral of motion:

$$E = \frac{1}{2} \dot{q}^2 + V(q)$$

and solution to the equation can be presented in the form of quadrature

$$t = \int \frac{dq}{\sqrt{2(E - V(q))}}$$

If $V(q)$ is a quartic polynomial, then the last integral can be represented in terms of elliptic functions [14], [15], [16].

It follows that the solution to the equation for the anharmonic oscillator (the Duffing equation)

$$\ddot{q}(t) + \omega^2 q(t) = -\epsilon q^3(t), \quad \omega > 0, \quad \epsilon > 0 \quad (3.1)$$

has the representation

$$q(t) = a \operatorname{cn}(\Omega t + b, k), \quad (3.2)$$

where $\text{cn}(u, k)$ is the elliptic cosine function of argument u and modulus k , a is the amplitude and b is the phase. The frequency Ω and the modulus of elliptic function k are obtained from parameters of the equation ω, ϵ and depend on amplitude a :

$$\Omega = \sqrt{\omega^2 + \epsilon a^2}, \quad k = \sqrt{\frac{\epsilon a^2}{2(\omega^2 + \epsilon^2)}}. \quad (3.3)$$

The function $\text{cn}(\Omega t + b, k)$ can be expanded in trigonometric series

$$\text{cn}(\Omega t + b, k) = \frac{2\pi}{k\mathbf{K}} \sum_{n=1}^{\infty} \frac{\mathbf{q}^{n-1/2}}{1 + \mathbf{q}^{2n-1}} \cos\left((2n-1)\frac{\pi(\Omega t + b)}{2\mathbf{K}}\right), \quad (3.4)$$

where

$$\mathbf{q} = e^{-\pi \frac{\mathbf{K}'}{\mathbf{K}}}, \quad \mathbf{q}' = e^{-\pi \frac{\mathbf{K}}{\mathbf{K}'}}. \quad (3.5)$$

Here $\mathbf{K} = \mathbf{K}(k)$ is the complete elliptic integral of the first kind and $\mathbf{K}' = \mathbf{K}(k')$, where $k^2 + k'^2 = 1$ [14], [15], [16]. In particular, with $b = -\mathbf{K}$ one obtains:

$$\text{cn}(\Omega t - \mathbf{K}, k) = \frac{2\pi}{k\mathbf{K}} \sum_{n=1}^{\infty} \frac{\mathbf{q}^{n-1/2}}{1 + \mathbf{q}^{2n-1}} \sin\left((2n-1)\frac{\pi\Omega t}{2\mathbf{K}}\right) \quad (3.6)$$

From this representation one can obtain an asymptotic expansion for small ϵ :

$$q(t) = A \sin\left(\omega\left(1 + \epsilon \frac{3A^2}{8\omega^2}\right)t\right) - \epsilon \frac{A^3}{32\omega^2} \sin\left(3\omega\left(1 + \epsilon \frac{3A^2}{8\omega^2}\right)t\right) + \dots, \quad (3.7)$$

here A is an arbitrary real parameter. This expansion one can also obtain by means of the Krylov-Bogoliubov averaging method [2].

3.2 Solution to the Higgs equation

Solution to the equation

$$\ddot{q} - \mu^2 q = -\epsilon q^3, \quad \epsilon > 0, \quad (3.8)$$

can belong to one of the three types depending on initial conditions

$$q(0) = q_0, \quad \dot{q}(0) = v_0. \quad (3.9)$$

A type of the solution is defined by energy

$$E = \frac{1}{2}v_0^2 - \frac{1}{2}\mu^2 q_0^2 + \frac{1}{4}\epsilon q_0^4 \quad (3.10)$$

Solutions have one of the following forms

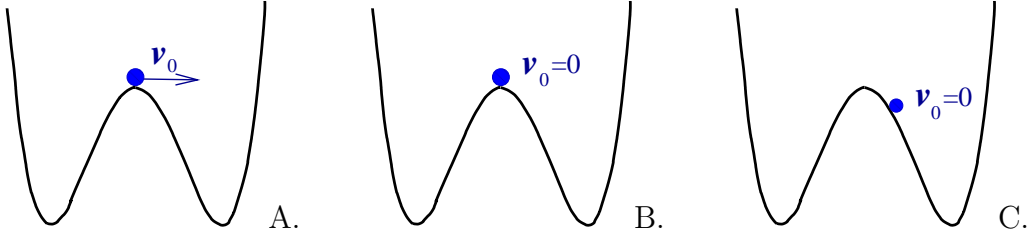


Figure 1: Potential energy and initial conditions. . The initial condition corresponding to a movement in two holes, $E > 0$; B. The initial condition, corresponding to a movement in two holes with infinite period, $E = 0$; C. The initial condition corresponding to a movement in one hole, $E < 0$.

- $E > 0$; the two-holes solutions (Fig 1A)

$$q(t) = a \operatorname{cn}(\Omega t + b, k) \quad (3.11)$$

$$a^2 = \frac{\mu^2}{\epsilon} \left(1 + \sqrt{1 + \frac{4\epsilon E}{\mu^4}} \right) \quad (3.12)$$

$$\Omega^2 = \mu^2 \sqrt{1 + \frac{4\epsilon E}{\mu^4}} \quad (3.13)$$

$$k^2 = \frac{1}{2} + \frac{1}{2} \frac{1}{\sqrt{1 + \frac{4\epsilon E}{\mu^4}}} \quad (3.14)$$

$$k'^2 = \frac{1}{2} - \frac{1}{2} \frac{1}{\sqrt{1 + \frac{4\epsilon E}{\mu^4}}} \quad (3.15)$$

Parameter b is defined from the condition

$$q(0) = a \operatorname{cn}(b, k). \quad (3.16)$$

We will discuss the case with initial data of the special form $q(0) = 0$. In this case

$$b = -\mathbf{K} \quad (3.17)$$

and

$$q(t) = a \operatorname{cn}(\Omega t - \mathbf{K}, k).$$

For this solution

$$v = \dot{q}(0) = \frac{a\mu k'}{\sqrt{1 - 2k'^2}}.$$

The energy E is expressed in terms of parameter k' ,

$$\frac{E\epsilon}{\mu^4} = \frac{\mu^4 k'^2 (1 - k'^2)}{(2k'^2 - 1)^2}. \quad (3.18)$$

We also note the dependency between a , ϵ and k'

$$a^2 = \frac{\mu^2}{\epsilon} \left(1 + \frac{1}{1 - 2k'^2} \right) \quad (3.19)$$

and Ω with μ and k'

$$\Omega^2 = \frac{\mu^2}{1 - 2k'^2}. \quad (3.20)$$

- $E = 0$; the two-holes solution with period $T = \infty$

$$q(t) = \mu \sqrt{\frac{2}{\epsilon}} \frac{1}{\cosh(t\mu + b)} \quad (3.21)$$

Parameter b is defined by initial condition

$$q(0) = \mu \sqrt{\frac{2}{\epsilon}} \frac{1}{\cosh(b)} \quad (3.22)$$

- $E < 0$; the one-hole solution, for instance, the solution in the right one

$$q(t) = a \operatorname{dn}(\Omega t + b, k) \quad (3.23)$$

$$a^2 = \frac{\mu^2}{\epsilon} \left(1 + \sqrt{1 + \frac{4\epsilon E}{\mu^4}} \right) \quad (3.24)$$

$$\Omega^2 = a^2 \frac{\epsilon}{2} = \frac{\mu^2}{2} \left(1 + \sqrt{1 + \frac{4\epsilon E}{\mu^4}} \right) \quad (3.25)$$

$$k^2 = 2 - \frac{2\mu^2}{\epsilon a^2} = 2 - \frac{2}{\left(1 + \sqrt{1 + \frac{4\epsilon E}{\mu^4}} \right)} \quad (3.26)$$

Parameter b is defined by the relation

$$q(0) = a \operatorname{dn}(b, k) \quad (3.27)$$

In the case with initial data of the special form $v_0 = 0$ we have

$$q(t) = a \operatorname{dn}(\Omega t - \mathbf{K}, k). \quad (3.28)$$

The rolling mode corresponds to a movement in the vicinity of the turning point, that is closest to the top of the potential (see Fig.1 C).

4. Elliptic functions expansion

We consider solution (3.18). This solution is periodical with period $T = 2\mathbf{K}/\Omega$. Here $\mathbf{K} = \mathbf{K}(k)$. We show that for t , such that $|t| < T/2$, the solution can be expanded in hyperbolic series (2.19). We also show that the first terms of the expansion (2.13) do represent an approximation to the exact solution (3.11) with $b = -\mathbf{K}$.

This remark is of importance because in a case of more complicated potentials or in a case of a problem with the friction we can explicitly apply the hyperbolic analogue of the Krylov-Bogoliubov method, while the exact solutions remain unknown, see, for instance, the problem with friction, (2.20).

We need an expansion of elliptic functions about the point that is equal to minus half-period, i.e. $\text{cn}(u - \mathbf{K}, k)$, for small k' . In this case $\mathbf{K} \rightarrow \infty$ as

$$\mathbf{K} \approx \ln \frac{4}{k'} + \frac{k'^2}{4} \ln \frac{4}{k'^2} - \frac{k'^2}{4} + \dots \quad (4.1)$$

According to (3.15) if $\epsilon \rightarrow 0$, then $k' \rightarrow 0$ and vice versa,

$$k' = \frac{\sqrt{E\epsilon}}{\mu^2} \left(1 - \frac{3}{2} \frac{E\epsilon}{\mu^4} + \mathcal{O}(\epsilon^2) \right) \quad (4.2)$$

The summation formula for the elliptic functions together with

$$\text{sn}(\mathbf{K}) = 1, \quad \text{cn}(\mathbf{K}) = 0, \quad \text{dn}(\mathbf{K}) = k' \quad (4.3)$$

gives

$$\begin{aligned} \text{cn}(u - \mathbf{K}) &= \frac{\text{cn}(u)\text{cn}(\mathbf{K}) + \text{sn}(u)\text{sn}(\mathbf{K})\text{dn}(u)\text{dn}(\mathbf{K})}{1 - k^2\text{sn}^2(u)\text{sn}^2(\mathbf{K})} \\ &= \frac{\text{sn}(u)\text{dn}(u)k'}{1 - k^2\text{sn}^2(u)} = \frac{\text{sn}(u)\text{dn}(u)k'}{\text{dn}^2(u)} = k' \frac{\text{sn}(u)}{\text{dn}(u)} \end{aligned} \quad (4.4)$$

4.1 Perturbation theory

In this section we investigate an expansion of the solution to (2.5) in terms of the elliptic function in the case of small parameter k' (i.e. ϵ) and compare the first terms of the series with the exact solution.

For small k' the following expansions take place [15], p.386, equations 16.15.2-16.15-3,

$$\text{sn}(u, k) = \tanh u + \frac{1}{4}k'^2 (\sinh u \cosh u - u) \frac{1}{\cosh^2 u} + \dots \quad (4.5)$$

$$\text{dn}(u, k) = \frac{1}{\cosh u} + \frac{1}{4}k'^2 (\sinh u \cosh u + u) \frac{\sinh u}{\cosh^2 u} + \dots \quad (4.6)$$

Consequently, for small k' we have

$$\begin{aligned} \frac{\text{sn}(u, k)}{\text{dn}(u, k)} &= \sinh u + \frac{1}{4}k'^2 (\sinh u - \sinh^2 u) \\ &\quad - \frac{1}{4}k'^2 u \left(\frac{1}{\cosh u} - \tanh u \right) + \dots \end{aligned} \quad (4.7)$$

and we get the following expansion

$$\text{cn}(u - \mathbf{K}, k) = k' \mathbf{c}_0(u) + k'^3 \mathbf{c}_1(u) + \dots \quad (4.8)$$

where

$$\mathbf{c}_0(u) = \sinh u \quad (4.9)$$

$$\mathbf{c}_1(u) = \frac{1}{4} (\sinh u - \sinh^2 u) - \frac{1}{4} u \left(\frac{1}{\cosh u} - \tanh u \right) \quad (4.10)$$

Expansion (4.10) contains a secular term $k'^3 u$, however, unlike trigonometric case, it is dominated by the term $\frac{1}{4}k'^3 \sinh^2 u$ with large u . First two terms of the expansion (4.10) are presented in Fig.2 A. Modified perturbation theory will be developed using an expansion in hyperbolic functions in the next subsection. This modified theory gives a better approximation to the exact solution for large u . It is interesting to mention, that, as it will be shown in the next subsection in the modified perturbation theory the term $-\frac{1}{16}k'^3 \sinh 3u$ contributes greatly. Its contribution increases faster with increase of u , than the term from perturbation theory $\frac{1}{4}k'^3 \sinh^2 u$.

In the Fig. 2 the comparison of the first two terms of perturbation theory with exact solution is presented. The first approximation is the upper solid line, the second one is plotted by the dotted line. The lower thick line shows the exact solution. It can be seen that the first and the second approximations of conventional perturbation theory almost coincide with each other. The second approximation calculated according to the conventional perturbation theory is not in fact an improvement of the first one.

4.2 Modified perturbation theory

We will show that the function $\text{cn}(u - \mathbf{K}, k)$, with $|u| < \mathbf{K}$ and small k' , allows the following representation

$$\text{cn}(u - \mathbf{K}, k) = \sum_{n=0}^{\infty} (-1)^n \mathcal{A}_n \sinh((2n+2)u') \quad (4.11)$$

where

$$\rho' = \frac{\pi \mathbf{K}}{2\mathbf{K}'}, \quad u' = \frac{\pi u}{2\mathbf{K}'} \quad (4.12)$$

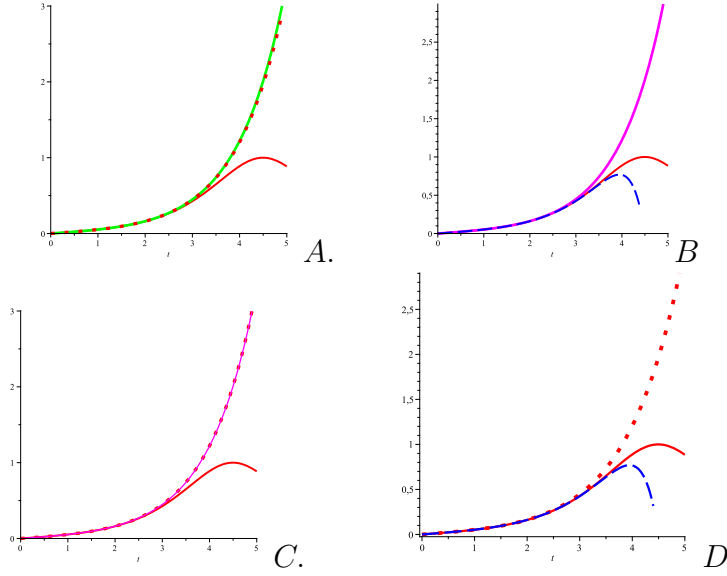


Figure 2: The first two approximations comparison, calculated according to conventional perturbation theory, (4.9) and (4.10), with the first two approximations of the modified perturbation theory, (4.15) and (4.16), and with the exact expression $cn(u - \mathbf{K}, k)$, $k = 0.999$.

$$\mathcal{A}_n \equiv \frac{\pi}{k\mathbf{K}' \cosh} \frac{1}{(2n+1)\rho'} \quad (4.13)$$

and the series (4.11) converges with $|\Re u| < \mathbf{K}$ and $-1 < k < 1$.

In order to prove (4.11) we use (4.4). Note that for $\frac{\text{sn}(u)}{\text{dn}(u)}$ the following representation holds, see equation (133), p. 85, [14],

$$\frac{\text{sn}(u)}{\text{dn}(u)} = \frac{\pi}{kk'\mathbf{K}'} \left\{ \frac{\sinh u'}{\cosh \rho'} - \frac{\sinh 3u'}{\cosh 3\rho'} + \frac{\sinh 5\rho'}{\cosh 5u'} + \dots \right\} \quad (4.14)$$

multiplication of (4.14) by k' gives (4.11).

We will discuss approximations

$$\mathfrak{E}^{(0)}(u) = \mathcal{A} \sinh u', \quad (4.15)$$

$$\mathfrak{E}^{(2)}(u) = \mathcal{A} \sinh u' - \mathcal{A}_1 \sinh 3u', \quad (4.16)$$

$$\mathfrak{E}^{(n)}(u) = \mathcal{A} \sinh u' - \mathcal{A}_1 \sinh 3u' + \dots + (-1)^n \mathcal{A}_n \sinh(2n+1)u' \quad (4.17)$$

Within our notation $\mathcal{A} \equiv \mathcal{A}_0$. We note, that u' and $\mathcal{A}_{2n+1}, n = 0, 1, 2$, have the

following expansions

$$u' = \frac{\pi}{2\mathbf{K}'} u \approx \left(1 - \frac{1}{4}k'^2 + \dots\right) u \quad (4.18)$$

$$\mathcal{A} = k' + \frac{7}{16}k'^3 + \frac{79}{256}k'^5 + O(k'^7) \quad (4.19)$$

$$\mathcal{A}_1 = -\left(\frac{1}{16}k'^3 + \frac{1}{16}k'^5 + O(k'^7)\right) \quad (4.20)$$

$$\mathcal{A}_3 = \frac{1}{4096}k'^7 + O(k'^9) \quad (4.21)$$

However with our notations we do not expand \mathcal{A}_i and u' in (4.18) - (4.21) in power series of k'^2 .

Approximations that are truncated series expansion of (4.11), namely approximations (4.15), (4.16) are shown in Fig.2. B. A comparison of these approximations with ones obtained within the conventional perturbation theory are presented in Fig.2.C and 2.D. In Fig.2.B the first two approximations calculated according to the modified perturbation theory are shown. It can be seen that the second approximation is an improvement of the first one. A comparison of the first approximations calculated according to conventional and modified perturbation theories are presented in Fig.2.C. These approximations almost coincide with each other. However, the second approximation calculated according to the conventional perturbation theory and the second approximation calculated according to the modified perturbation theory differ significantly as it can be seen in Fig.2.D. The second approximation calculated according to the modified perturbation theory is closer to the exact solution in comparison with the second approximation obtained according to the conventional perturbation theory. Next terms of the expansion are shown in Fig.3.

4.3 Bounds for elliptic cosine expansion

Let us prove the following lemma.

Lemma 1 *Given $\mu > 0$ and $0 < c < 1$. Than the following inequality holds*

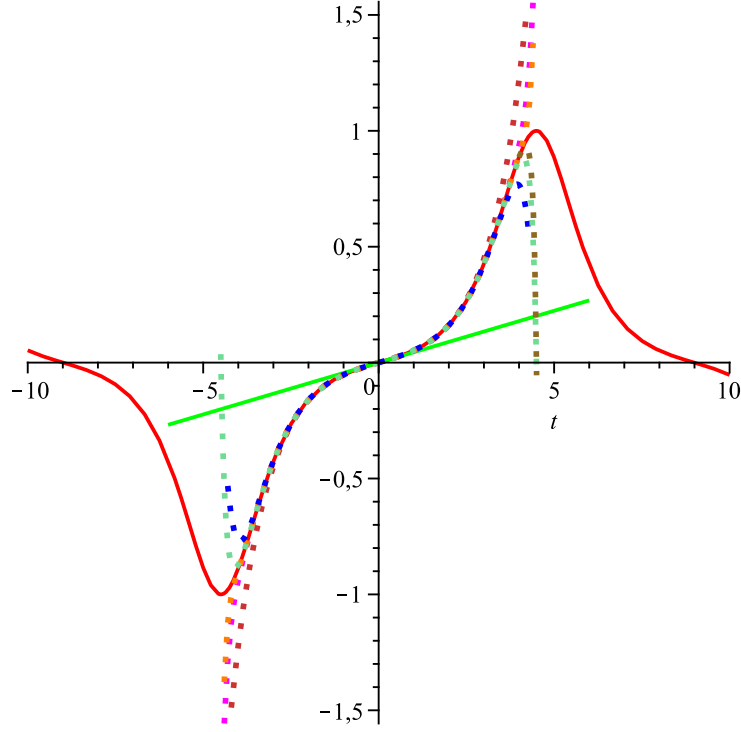
$$\left| \text{cn}(\Omega t - \mathbf{K}, k) - \mathfrak{C}^{(n-1)}(\Omega t) \right| \leq \frac{4}{k} (k')^{(2n+1)(1-c)}, \quad n = 1, 2, \dots \quad (4.22)$$

for all

$$0 \leq t \leq \frac{c}{\mu\sqrt{2}} \ln \frac{1}{k'}, \quad 0 \leq k' \leq 1/2 \quad (4.23)$$

here $\mathfrak{C}^{(n)}(u)$ stays for the approximation defined by (4.17).

Remark 1 *Estimation (4.22) is not trivial in comparison to the proposition that the series (4.11) converges with $|\Re u| < \mathbf{K}$ and $-1 < k < 1$. The reason for that is that the estimate (4.22) according to (A.19) allows to consider $t \rightarrow \infty$ with $k' \rightarrow 0$.*



A.

Figure 3: (color online) The function of $t \operatorname{cn}(t - \mathbf{K}, k)$ is shown with the red line with $k = 0.999$. The green line depicts a linear approximation. The orange, purple and coral dashed lines depict the first, the third and the fifth approximations calculated by formula (4.11) respectively dotted lines of cyan, aquamarine and ochre color depict approximations of the second, the fourth and the sixth orders calculated by (4.11).

Proof is based on the estimation of the residual term

$$r_n \equiv \frac{\pi}{k\mathbf{K}'} \left\{ -\frac{\sinh((2n+1)\rho'u'')}{\cosh((2n+1)\rho')} + \frac{\sinh((2n+3)\rho'u'')}{\cosh((2n+3)\rho')} + \dots \right\} \quad (4.24)$$

by the geometric sequence sum

$$d_n \equiv \frac{\pi}{k\mathbf{K}'} \frac{e^{-(2n+1)\rho'(1-u'')}}{1 - e^{-2\rho'(1-u'')}} \quad (4.25)$$

that is,

$$|r_n| \leq d_n, \quad (4.26)$$

Taking into account

$$e^{-2\rho'(1-u'')} = \exp\left(-2\frac{\pi\mathbf{K}}{2\mathbf{K}'}\right) \exp\left(\frac{2\pi\Omega t}{2\mathbf{K}'}\right) \quad (4.27)$$

the estimation (4.26) makes if

$$\exp\left(-\frac{\pi\mathbf{K}}{2\mathbf{K}'}\right) \exp\left(\frac{\pi\Omega t}{2\mathbf{K}'}\right) < 1 \quad (4.28)$$

If this condition holds then

$$d_n \leq \frac{4}{k} \left(k'\right)^{2n+1} e^{(2n+1)\Omega t} < \frac{4}{k} (k')^{(2n+1)(1-c)} \rightarrow 0 \quad (4.29)$$

$$\text{for } k' \rightarrow 0 \text{ and } c < 1 \quad (4.30)$$

The condition (4.28) is satisfied if there is a restriction on t , such that

$$\left(k'\right)^2 \exp(2\Omega t) < \frac{1}{2} \quad (4.31)$$

Taking into account the latter limitation and the property of the elliptic functions formulated below the condition (4.28) we get

$$\exp\left(-\frac{\pi \mathbf{K}}{2\mathbf{K}'}\right) \exp\left(\frac{\pi \Omega t}{2\mathbf{K}'}\right) < k' e^{\Omega t} \quad (4.32)$$

The detailed proof can be found in Appendix A.

5. n -mode approximation

In the present section approximations of exact solutions are presented and accuracy of these approximations is estimated for small coupling constant ϵ .

Taking into account the series (4.11) let us consider an approximation of the exact solution (3.18). We get

$$q(t) = a \operatorname{cn}(\Omega t - \mathbf{K}, k) = \sum_{n=0}^{\infty} s_n(t) \quad (5.1)$$

where the following notations are introduced

$$s_0(t) = A \sinh\left(\frac{\pi \Omega t}{2\mathbf{K}'}\right), \quad s_n(t) = (-1)^n A_n \sinh\left(\frac{(2n+1)\pi \Omega t}{2\mathbf{K}'}\right), \quad n = 1, \dots \quad (5.2)$$

$$A = a \frac{\pi}{k\mathbf{K}'} \frac{1}{\cosh\left(\frac{\pi \mathbf{K}}{2\mathbf{K}'}\right)}, \quad A_n = a \frac{\pi}{k\mathbf{K}'} \frac{1}{\cosh\left(\frac{(2n+1)\pi \mathbf{K}}{2\mathbf{K}'}\right)}, \quad n = 1, \dots \quad (5.3)$$

Let us introduce the notion of n -order mode approximations

$$q_0(t) = s_0(t) \quad (5.4)$$

$$q_1(t) = s_0(t) + s_1(t) \quad (5.5)$$

$$q_2(t) = s_0(t) + s_1(t) + s_2(t) \quad (5.6)$$

...

$$q_n(t) = s_0(t) + s_1(t) + \dots + s_n(t) \quad (5.7)$$

We note that arguments of the sinh functions and A_i , $i = 1, \dots$ are power series of k'^2 , however in definitions (5.4) - (5.7) we do make these expansions.

In order to estimate an error of approximation of (5.4) - (5.7) the following estimations for $k' < 1/2$ are used. The following Lemma provides the estimations

Lemma 2 *For $k' < 1/2$ the following estimations hold*

1.

$$a = \frac{\mu}{\sqrt{\epsilon}} \left(1 + \frac{1}{1 - 2k'^2} \right)^{1/2} < \frac{2\mu}{\sqrt{\epsilon}} \quad (5.8)$$

2.

$$k'^2 = \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 + \frac{4\epsilon E}{\mu^4}}} \right) \leq \frac{2\epsilon E}{\mu^4} \quad (5.9)$$

3.

$$\Omega = \mu \left(1 + \frac{4\epsilon E}{\mu^4} \right)^{1/4} < \mu \left(1 + \frac{4\epsilon E}{\mu^4} \right) \quad (5.10)$$

Proof (5.8) follows from the fact that if $k' < 1/2$, then (see. Fig. 4. A)

$$\frac{1}{1 - 2k'^2} < 2 \quad (5.11)$$

Taking into account (3.19) we get (5.8).

The estimation (5.9) we get in the following way. The estimate holds, see Fig. 4 B.,

$$\frac{1}{2y} \left(1 - \frac{1}{\sqrt{1 + y}} \right) < \frac{1}{2}, \quad 0 \leq y < \infty \quad (5.12)$$

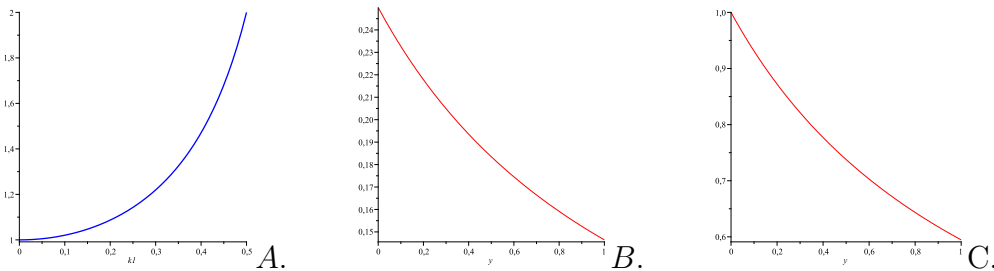


Figure 4: A. The estimate (5.12) is shown in the plot: k'^2 is along horizontal axis, and the left-hand side of (5.12) is on the vertical one. B. The illustration of the estimate (5.13). y is along the horizontal axis, and the left-hand side of (5.13). along the vertical one. C. The illustration of the estimate (5.15). y is along the horizontal axis, and the left-hand side of (5.15) is on the vertical axis.

Therefore

$$k'^2 = \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 + \frac{4\epsilon E}{\mu^4}}} \right) \leq \frac{2\epsilon E}{\mu^4} \quad (5.13)$$

The estimate

$$\frac{(1+y)^{1/4}}{1+y} \leq 1 \quad (5.14)$$

is shown in Fig.4 C. We get

$$\Omega = \mu \left(1 + \frac{4\epsilon E}{\mu^4}\right)^{1/4} < \mu \left(1 + \frac{4\epsilon E}{\mu^4}\right) \quad (5.15)$$

From the estimations (5.9), (A.21) and (5.8) the estimation of the exact solution follows (5.4)

$$|q(t) - q_1(t)| \leq ad_1 \leq 2\mu \sqrt{\frac{1}{\epsilon} \frac{4}{k}} \left(\frac{2\epsilon E}{\mu^4}\right)^{3(1-c)/2} = \frac{8\sqrt{2E}}{k\mu} \left(\frac{2\epsilon E}{\mu^4}\right)^{1-\frac{3}{2}c}, \quad (5.16)$$

for

$$0 \leq t < \frac{c}{\mu 2\sqrt{2}} \ln \frac{1}{k'^2}, \quad c < 1/2. \quad (5.17)$$

Let us note that according to (5.9)

$$k'^2 \leq \frac{2\epsilon E}{\mu^4} \quad (5.18)$$

one has

$$\frac{1}{k'^2} \geq \frac{\mu^4}{2\epsilon E} \Rightarrow \ln \frac{1}{k'^2} \geq \ln \frac{\mu^4}{2\epsilon E} \quad (5.19)$$

i.e. if the following is true

$$t < \frac{c}{\mu 2\sqrt{2}} \ln \frac{\mu^4}{2\epsilon E} \quad (5.20)$$

then (5.17) holds.

In the same way we get

Lemma 3 For

$$t < \frac{c}{\mu 2\sqrt{2}} \ln \frac{\mu^4}{2\epsilon E} \quad (5.21)$$

estimations

$$|q(t) - q_{n-1}(t)| \leq \frac{8\sqrt{2E}}{k\mu} \left(\frac{2\epsilon E}{\mu^4}\right)^{n-\frac{2n+1}{2}c}, \quad n = 1, \dots \quad (5.22)$$

hold.

Proof follows from (A.25) and estimations

$$\begin{aligned} |q(t) - q_{n-1}(t)| &\leq \frac{4a}{k} \left(k'\right)^{(2n+1)} e^{(2n+1)\Omega t} < a \frac{4}{k} (k')^{(2n+1)(1-c)} \\ &= \frac{8\sqrt{2E}}{k\mu} \left(\frac{2\epsilon E}{\mu^4}\right)^{n-\frac{2n+1}{2}c} \end{aligned} \quad (5.23)$$

6. Rearrangement of the series for the elliptic cosine

The following presentation allows to obtain the series (2.13) from (5.1). Particularly, the solution (5.1) can be presented in the following form:

$$q(t) = a \operatorname{cn}(\Omega t - \mathbf{K}, k) = \sum_{n=0}^{\infty} (-1)^n A \bar{a}_n(\lambda) \sinh((2n+1)\mu \bar{\Omega}(\lambda) t) \quad (6.1)$$

Here

$$\lambda = \frac{A^2 \epsilon}{\mu^2}, \quad (6.2)$$

and $\bar{a}_n(\lambda)$ and $\bar{\Omega}(\lambda)$ are given by functions $a_n(k'^2)$ and $\Omega(k'^2)$

$$a_n(k'^2) = \frac{\cosh\left(\frac{\pi \mathbf{K}}{2\mathbf{K}'}\right)}{\cosh\left(\frac{(2n+1)\pi \mathbf{K}}{2\mathbf{K}'}\right)}, \quad n = 1, \dots; \quad a_0(\lambda) = 1, \quad (6.3)$$

$$\Omega(k'^2) = \frac{1}{\mu} \frac{\pi \Omega}{2\mathbf{K}'} = \frac{1}{\sqrt{1-2k'^2}} \frac{\pi}{2\mathbf{K}'}. \quad (6.4)$$

and via change of variables, i.e.

$$\bar{a}_n(\lambda) = a_n(F^{-1}(\lambda)), \quad n = 1, \dots; \quad a_0(\lambda) = 1, \quad (6.5)$$

$$\bar{\Omega}(\lambda) = \Omega(F^{-1}(\lambda)). \quad (6.6)$$

F^{-1} is defined from the relation

$$\lambda = F(k'^2), \quad F(k'^2) = \left(1 + \frac{1}{1-2k'^2}\right) \left(\frac{\frac{\pi}{k\mathbf{K}'}}{\cosh\left(\frac{\pi \mathbf{K}}{2\mathbf{K}'}\right)}\right)^2 \quad (6.7)$$

For the lowest terms we have

$$\lambda = 2k'^2 + \frac{15}{4}k'^4 + \mathcal{O}(k'^6). \quad (6.8)$$

The proof is obvious. The not so obvious fact is that the series calculated in such a way coincides with the series (2.13), see Lemma 4 below.

6.1 n -mode approximation with power series of k'

Consider the following power series of k'^2

$$\Omega = \frac{1}{\sqrt{1-2k'^2}} \frac{\pi}{2\mathbf{K}'} = 1 + k'^2 \Omega_1 + k'^4 \Omega_2 + \dots, \quad (6.9)$$

$$\frac{\cosh\left(\frac{\pi \mathbf{K}}{2\mathbf{K}'}\right)}{\cosh\left((2i+1)\frac{\pi \mathbf{K}}{2\mathbf{K}'}\right)} = k'^{2i} l_{i0} + k'^{2(i+1)} l_{i1} + \dots \quad (6.10)$$

Explicit form of its lowest terms is

$$\Omega_1 = \frac{3}{4}, \quad \Omega_2 = \frac{75}{64}, \quad l_{10} = \frac{1}{16}, \quad l_{11} = \frac{9}{256}. \quad (6.11)$$

Using approximation

$$\Omega^{(n)} = (1 + k'^2 \Omega_1 + \dots k'^{2n} \Omega_n), \quad (6.12)$$

$$A_i^{(n)} = A (k'^{2i} l_{i0} + k'^{2(i+1)} l_{i1} + \dots + k'^{2(i+n)} l_{in}). \quad (6.13)$$

we will obtain the following approximations for each term in the series (5.1)

$$s_0^{(0)}(t) = A \sinh(\mu t), \quad (6.14)$$

$$s_0^{(1)}(t) = A \sinh((1 + k'^2 \Omega_1) \mu t), \quad (6.15)$$

$$s_0^{(n)}(t) = A \sinh((1 + k'^2 \Omega_1 + \dots k'^{2n} \Omega_n) \mu t). \quad (6.16)$$

We notice that, with adopted notations we do not expand A in power series of k'^2 .

In the same way,

$$s_i^{(n,j)}(t) = (-1)^i A_i^{(n)} \sinh((2i+1) \mu \Omega^{(j)} t), \quad (6.17)$$

or explicitly

$$s_i^{(n,j)}(t) = (-1)^i A (k'^{2i} l_{i,0} + \dots + k'^{2(n+i)} l_{in}) \sinh((2i+1) \mu (1 + k'^2 \Omega_1 + \dots k'^j \Omega_j) t) \quad (6.18)$$

In accordance with (6.9) and (6.13) we have the following approximations for $s_1(t)$

$$s_1^{(0,0)}(t) = -A k'^2 l_{1,0} \sinh(3 \mu t), \quad (6.19)$$

$$s_1^{(0,1)}(t) = -A k'^2 l_{1,0} \sinh(3 \mu (1 + k'^2 \Omega_1) t), \quad (6.20)$$

$$s_1^{(1,2)}(t) = -A (k'^2 l_{1,0} + k'^4 l_{11}) \sinh(3 \mu (1 + k'^2 \Omega_1 + k'^4 \Omega_2) t). \quad (6.21)$$

We will also use notations:

$$s_i^{(exact,j)}(t) = (-1)^i A_i \sinh((2i+1) \mu (1 + k'^2 \Omega_1 + \dots + k'^{2j} \Omega_j) t). \quad (6.22)$$

$$s_i^{(n,exact)}(t) = (-1)^i A (k'^{2i} l_{i,0} + \dots + k'^{2(n+i)} l_{in}) \sinh((2i+1) \mu \Omega t). \quad (6.23)$$

6.2 n -mode approximation with power series of λ

In order to express the series (5.1) in a way analogous to (2.13) we have to expand Ω and A_i/A in power series at ϵ

$$\Omega = \mu \left[1 + \frac{A^2 \epsilon}{\mu^2} \bar{\Omega}_1 + \left(\frac{A^2 \epsilon}{\mu^2} \right)^2 \bar{\Omega}_2 + \dots \right] \quad (6.24)$$

$$A_i = A \left[\left(\frac{A^2 \epsilon}{\mu^2} \right)^i \bar{l}_{i0} + \left(\frac{A^2 \epsilon}{\mu^2} \right)^{i+1} \bar{l}_{i1} + \dots \right] \quad (6.25)$$

$$\bar{\Omega}_1 = \frac{3}{8}, \quad \bar{l}_1 = -\frac{1}{32}. \quad (6.26)$$

We will also use notations

$$\Omega^{(n)} = \mu \left[1 + \frac{A^2 \epsilon}{\mu^2} \bar{\Omega}_1 + \left(\frac{A^2 \epsilon}{\mu^2} \right)^2 \bar{\Omega}_2 + \dots + \left(\frac{A^2 \epsilon}{\mu^2} \right)^n \bar{\Omega}_n \right] \quad (6.27)$$

$$A_i^{(n)} = A \left[\left(\frac{A^2 \epsilon}{\mu^2} \right)^i \bar{l}_i + \left(\frac{A^2 \epsilon}{\mu^2} \right)^{i+1} \bar{l}_{i1} + \dots + \left(\frac{A^2 \epsilon}{\mu^2} \right)^{i+n} \bar{l}_{in} \right] \quad (6.28)$$

$$(6.29)$$

In the lowest terms we have the following approximations

$$\bar{s}_1^{(0)}(t) = A \sinh(\mu t) \quad (6.30)$$

$$\bar{s}_0^{(1)}(t) = A \sinh \left(\left(1 + \frac{A^2 \epsilon}{\mu^2} \bar{\Omega}_1 \right) \mu t \right) \quad (6.31)$$

$$\bar{s}_1^{(0,1)}(t) = -\frac{A^3 \epsilon}{\mu^2} \bar{l}_1 \sinh \left(3 \left(1 + \frac{A^2 \epsilon}{\mu^2} \bar{\Omega}_1 \right) \mu t \right) \quad (6.32)$$

$$\begin{aligned} \bar{s}_1^{(1,2)}(t) = & -A \left(\frac{A^2 \epsilon}{\mu^2} \bar{l}_1 + \left(\frac{A^2 \epsilon}{\mu^2} \right)^2 \bar{l}_2 \right) \\ & \sinh \left(3 \left(1 + \frac{A^2 \epsilon}{\mu^2} \bar{\Omega}_1 + \left(\frac{A^2 \epsilon}{\mu^2} \right)^2 \bar{\Omega}_2 \right) \mu t \right) \end{aligned} \quad (6.33)$$

here $\bar{\Omega}_1$ and \bar{l}_1 are given by (6.26). The bar over s_i stays for expansions in power series of ϵ (or λ).

We have the following Lemma

Lemma 4 *When amplitudes (5.3) and (2.19) are equaled, i.e.*

$$\mathfrak{A} = A \quad (6.34)$$

the following relations hold

$$\mathfrak{a}_{nk} = \bar{l}_{nk} \quad (6.35)$$

$$\mathfrak{m}_i = \bar{\Omega}_i \quad (6.36)$$

Lemma (4) allows to claim that if $\mathfrak{A} = A$ then

$$\mathfrak{s}_0^{(i)}(t) = \bar{s}_0^{(i)}(t), \quad i = 1, \dots \quad (6.37)$$

$$\mathfrak{s}_n^{(i,j)}(t) = \bar{s}_n^{(i,j)}(t) \quad n = 1, \dots, i, j = 0, 1, \dots \quad (6.38)$$

7. Bounds to the hyperbolic analogue of the modified perturbation theory

The goal of the discussions below is to obtain an approximation error estimate for (2.19) mentioned at the beginning of the paper. To do this we mention that in approximation (5.7) constant A_i and the sinh functions argument are the power series of k'^2 , and consequently the powers of ϵ .

7.1 One mode approximations

Approximations to the exact one mode expression

$$q_0(t) = A \sinh \left(\left(\frac{\pi \Omega}{2\mathbf{K}'} \right) t \right) \quad (7.1)$$

have the form

$$\bar{s}_0^{(0)}(t) = A \sinh(\mu t) \quad (7.2)$$

$$\bar{s}_0^{(1)}(t) = A \sinh \left(\mu \left(1 + \frac{3A^2}{8\mu^2} \epsilon t \right) \right) \quad (7.3)$$

$$\bar{s}_0^{(2)}(t) = A \sinh \left(\mu \left(1 + \frac{3A^2}{8\mu^2} \epsilon - \frac{15A^4}{2^8 \mu^4} \epsilon^2 \right) t \right) \quad (7.4)$$

In relations (7.2) - (7.4) the sinh-functions arguments are different from ones in (7.1). We denote these differences by Δ_n . (calculations in the file approx-ch.mw).

$$\Delta_0 = \left(\mu - \frac{\pi \Omega}{2\mathbf{K}'} \right) t \quad (7.5)$$

$$\Delta_1 = \left(\mu \left(1 + \frac{3A^2}{8\mu^2} \epsilon \right) - \frac{\pi \Omega}{2\mathbf{K}'} \right) t \quad (7.6)$$

$$\Delta_2 = \left(\mu \left(1 + \frac{3A^2}{8\mu^2} \epsilon - \frac{15A^4}{2^8 \mu^4} \epsilon^2 \right) - \frac{\pi \Omega}{2\mathbf{K}'} \right) t \quad (7.7)$$

Let us estimate contribution due to the mentioned differences. We have

$$\begin{aligned} s_0(t) - s_0^{(n)}(t) &= A [\sinh(\mu \Omega t) - \sinh(\mu \Omega t + \Delta^{(n)})] \\ &= A [\sinh(\mu \Omega t) - \sinh(\mu \Omega t) \cosh(\Delta_n) - \cosh(\mu \Omega t) \cdot \sinh(\Delta_n)] \\ &= A \sinh(\mu \Omega t) (1 - \cosh(\Delta_n)) - A \cosh(\mu \Omega t) \cdot \sinh(\Delta_n) \end{aligned} \quad (7.8)$$

Taking into account that for small Δ , for instance, $0 < \Delta < 1$, the following relation holds

$$\sinh(\Delta) < 2\Delta, \quad (7.9)$$

$$|1 - \cosh(\Delta)| < 2\Delta, \quad (7.10)$$

we get for $t > 0$ and

$$\Delta_n < 1, \quad (7.11)$$

$$\begin{aligned} |s_0(t) - s_0^{(n)}(t)| &< 2\Delta_n A \sinh(\mu \Omega t) + 2\Delta_n A \cosh(\mu \Omega t) \\ &= 2\Delta_n A e^{\mu \Omega t}, \end{aligned} \quad (7.12)$$

$$(7.13)$$

At this point one can see that the error of the approximation of $s_0^{(n)}(t)$ to $s_0(t)$ is defined by infinitesimality of Δ_n .

In particularly, for Δ_0 we have

$$\Delta_0 = \mu t \delta_0, \quad (7.14)$$

$$\delta_0 = 1 - \frac{\pi}{2\mathbf{K}'} \frac{1}{\sqrt{1 - 2k'^2}}. \quad (7.15)$$

From analysis (see Fig. 5) we get

$$\delta_0 < \sqrt{2}k'^2, \quad 0 \leq k' \leq \frac{1}{2}, \quad (7.16)$$

and therefore,

$$\Delta_0 < \sqrt{2}\mu k'^2 t, \quad 0 \leq k' \leq \frac{1}{2} \quad (7.17)$$

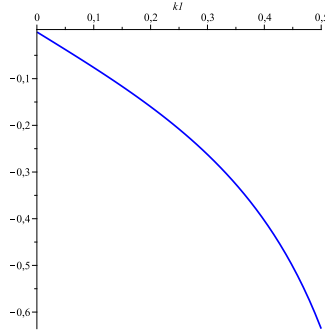


Figure 5: The estimate (7.15) is showed. Along the horizontal axis k'^2 are plotted, and on vertical axis the relation the relation of the right-hand side of (7.15) to k'^2 .

Estimates Δ_n , $n = 1, 2$ are presented in Appendix.B.

7.2 Two modes approximations

Consider the two modes term $q_1(t)$,

$$q_1(t) = q_0(t) + s_1(t), \quad (7.18)$$

where

$$s_1(t) = -A_1 \sinh\left(\frac{3\pi\Omega t}{2\mathbf{K}'}\right), \quad n = 1, \dots \quad (7.19)$$

$$A_1 = a \frac{\pi}{k\mathbf{K}'} \frac{1}{\cosh\left(\frac{3\pi\mathbf{K}}{2\mathbf{K}'}\right)} \quad (7.20)$$

and as an approximation to $s_1(t)$ take

$$\bar{s}_1^{(0,0)} = -\frac{\epsilon A^3}{32\mu^2} \sinh(3\mu t) \quad (7.21)$$

or

$$s_1^{(0,0)}(t) = -A k'^2 \frac{1}{16} \sinh(3\mu t) \quad (7.22)$$

A deviation of $\bar{s}_1^{(0,0)}$ from the exact $s_1(t)$ is not only due to changes in the argument of \sinh - function, $3\Delta_0$, but also due to corrections to the exact value of A_1 ,

$$A_1 = \frac{1}{16} k'^2 A + a k'^5 L_1(k') \quad (7.23)$$

and we have an estimation

$$L_1 < 0.1 \quad \text{for} \quad k' < 1/2. \quad (7.24)$$

This estimation is true since (see Fig. 6).

$$-\frac{k'^2}{16} \frac{\pi}{k\mathbf{K}'} \frac{1}{\cosh\left(\frac{\pi\mathbf{K}}{2\mathbf{K}'}\right)} + \frac{\pi}{k\mathbf{K}'} \frac{1}{\cosh\left(\frac{3\pi\mathbf{K}}{2\mathbf{K}'}\right)} < \frac{1}{10} k'^5. \quad (7.25)$$

We get

$$s_1^{(0,0)}(t) - s_1(t) = -A k'^2 \frac{1}{16} \sinh(3\mu t) + A_1 \sinh(3\mu\Omega t) \quad (7.26)$$

$$= -A k'^2 \frac{1}{16} \sinh(3\mu(\delta_0 + \Omega)t) + A_1 \sinh(3\mu\Omega t) \quad (7.27)$$

$$= -A k'^2 \frac{1}{16} (\sinh(3\mu(\delta_0 + \Omega)t) - \sinh(3\mu\Omega t)) \quad (7.28)$$

$$+ a \left(-\frac{k'^2}{16} A + A_1 \right) \sinh(3\mu\Omega t) \quad (7.29)$$

Consequently

$$|s_1^{(0,0)}(t) - s_1(t)| < -A k'^2 \frac{1}{16} |\sinh(3\mu(\delta_0 - \Omega)t) - \sinh(3\mu\Omega t)| \quad (7.30)$$

$$+ a \frac{k'^5}{10} |\sinh(3\mu\Omega t)| \quad (7.31)$$

Taking into account an estimation of the form (7.12) we get,

$$|s_1^{(0,0)}(t) - s_1(t)| < C k'^4 e^{3\mu\Omega t} \quad (7.32)$$

where C is constant.

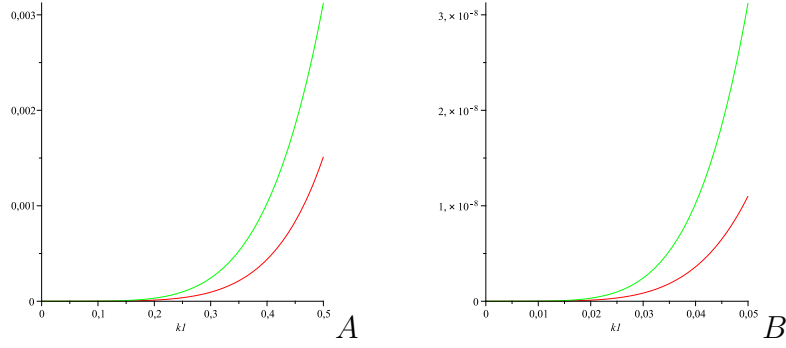


Figure 6: (color online) A. The estimation (7.25) is illustrated. On the horizontal axis k'^2 is plotted, on the vertical axis the values of the left-hand side of (7.25) (red color) and the right-hand side (7.25) (green color) are presented. B. The zoom of the plot in the left panel A for small values of k'^2 .

7.3 The second order of the modified perturbation theory approximation

The following theorem holds

Theorem 1 *For*

$$0 < t < \frac{c}{2\sqrt{2}\mu} \ln \frac{\mu^4}{2\epsilon E} \quad \text{and} \quad 0 < c < \frac{1}{2} \quad (7.33)$$

there are C and $\sigma < 3/4$ such that, the following estimation is true

$$|q(t) - s_0^{(1)}(t) - s_1^{(0,0)}| \leq C \frac{\sqrt{E}}{\mu} \left(\frac{\epsilon E}{\mu^4} \right)^{2-\sigma}, \quad n = 1, \dots \quad (7.34)$$

Remark. In the inequality (7.34) C – dimensionless constant, factor $\frac{\sqrt{E}}{\mu}$ is introduced to get dimensionless quantity, $\frac{\epsilon E}{\mu^4}$ – dimensionless factor.

Proof. It follows from the estimates (5.22) and (7.32). Indeed, from (5.22) it follows that

$$|q(t) - q_1(t)| \leq \frac{8\sqrt{2E}}{k\mu} \left(\frac{2\epsilon E}{\mu^4} \right)^{2-\frac{3}{2}c}, \quad n = 1, \dots \quad (7.35)$$

In accordance with (5.5)

$$q_1(t) = q_0(t) + s_1(t) \quad (7.36)$$

and in the estimation (7.34) we took the approximation $s_0^{(1)}(t)$ to $q_0(t)$ and the approximation $s_1^{(0,0)}$ to $s_1(t)$. We get

$$q(t) - s_0^{(1)}(t) - s_1^{(0,0)} = q(t) - q_1(t) + q_0(t) - s_0^{(1)}(t) + s_1(t) - s_1^{(0,0)}(t), \quad (7.37)$$

consequently

$$|q(t) - s_0^{(1)}(t) - s_1^{(0,0)}| < |q(t) - q_1(t)| + |q_0(t) - s_0^{(1)}(t)| + |s_1(t) - s_1^{(0,0)}(t)| \quad (7.38)$$

In order to finish the proof we need the estimates (7.11) and (B.9) for (B.3). Gathering these estimations we get (7.34).

7.4 Bound of the n -mode approximation

Consider an n -mode $s_n(t)$,

$$\begin{aligned} s_n(t) &= (-1)^n A_n \sinh\left(\frac{(2n+1)\pi\Omega t}{2\mathbf{K}'}\right) \\ &= (-1)^n A l_n \sinh((2n+1)\mu\Omega t), \quad n = 1, \dots \end{aligned} \quad (7.39)$$

where

$$\begin{aligned} A_n &= a \frac{\pi}{k\mathbf{K}'} \frac{1}{\cosh\left(\frac{(2n+1)\pi\mathbf{K}}{2\mathbf{K}'}\right)} \\ &= A \frac{\cosh\left(\frac{\pi\mathbf{K}}{2\mathbf{K}'}\right)}{\cosh\left(\frac{(2n+1)\pi\mathbf{K}}{2\mathbf{K}'}\right)} = \frac{A \cosh(\mathbf{K}\Omega)}{\cosh((2n+1)\mathbf{K}\Omega)} \end{aligned} \quad (7.40)$$

and

$$l_n(k') = \frac{\cosh(\mathbf{K}\Omega)}{\cosh((2n+1)\mathbf{K}\Omega)} \quad (7.41)$$

As an approximation to $s_n(t)$ we take

$$s_n^{(r,m)} = (-1)^n A l_n^{(r)} \sinh((2n+1)\mu\Omega^{(m)}t) \quad (7.42)$$

where the series $l_n^{(r)}$ is obtained from the expansion of the right-hand side of (7.41)

$$l_n^{(r)} = l_{n,0} k'^{2n} + \dots + l_{n,r} k'^{2(n+r)} \quad (7.43)$$

and $\Omega^{(m)}$ is defined in (6.9),

$$\Omega^{(m)} = 1 + \Omega_1 k'^2 + \dots + \Omega_m k'^{2m} \quad (7.44)$$

The following lemmas are true [18].

Lemma 5 For $0 < k' < 1/2$, $r \geq 1$, $n \geq 1$, there are constants $B^{(r)}$ and $L_n^{(r)}$ such that

$$|\Omega(k') - \Omega^{(r)}(k')| < B^{(r)} k'^{2(r+1)} \quad (7.45)$$

$$|l_n(k') - l_n^{(r)}(k')| < L_n^{(r)} k'^{2(n+r+1)}. \quad (7.46)$$

From Lemma 5 one gets the following

Lemma 6

$$|s_n(t) - s_n^{(r,m)}(t)| = S_n^{(r,m)} k'^{2(n+r+1)} e^{\sigma_n^{(r,m)} t} \quad (7.47)$$

7.5 Error of the n -mode approximation by power series of λ

The n -mode can be considered as a function of λ :

$$s_n(t) = (-1)^n A \bar{l}_n(\lambda) \sinh((2n+1)\mu \bar{\Omega}(\lambda)t), \quad n = 1, \dots \quad (7.48)$$

Using the power series on λ one introduces notations

$$\bar{s}_n^{(r,m)} = A \bar{l}_n^{(r)} \sinh((2n+1)\mu \bar{\Omega}^{(m)}t) \quad (7.49)$$

where $\bar{l}_n^{(r)}$ and $\bar{\Omega}^{(m)}$ are obtained via reexpansions of the right-hand side of (7.41) and (7.44) in a power series of λ

$$l_n^{(r)} = \bar{l}_{n,0}\lambda^n + \dots + \bar{l}_{n,r}\lambda^{(n+r)}, \quad (7.50)$$

$$\bar{\Omega}^{(m)} = 1 + \bar{\Omega}_1\lambda^2 + \dots + \bar{\Omega}_m\lambda^m \quad (7.51)$$

The following lemmas are true.

Lemma 7 *For λ small enough there are constant $\bar{B}^{(r)}$ and $\bar{L}_n^{(r)}$ such that*

$$|\bar{\Omega}(\lambda) - \bar{\Omega}^{(r)}(\lambda)| < \bar{B}^{(r)}\lambda^{(r+1)} \quad (7.52)$$

$$|\bar{l}_n(\lambda) - \bar{l}_n^{(r)}(\lambda)| < \bar{L}_n^{(r)}\lambda^{(n+r+1)}. \quad (7.53)$$

From Lemma 5 it follows that

Lemma 8

$$|\bar{s}_n(t) - \bar{s}_n^{(r,m)}(t)| = \bar{S}_n^{(r,m)}\lambda^{(n+r+1)}e^{\bar{\sigma}_n^{(r,m)}t} \quad (7.54)$$

The following theorem holds [18]:

Theorem 2 *For*

$$0 < t < \frac{c}{2\sqrt{2}\mu} \ln \frac{\mu^4}{2\epsilon E} \quad \text{and} \quad 0 < c < \frac{1}{2} \quad (7.55)$$

one can find constants C_n and $\sigma < 3n/4$ such that, the following estimate is true

$$|q(t) - q^{(n)}(t)| \leq C_n \frac{\sqrt{E}}{\mu} \left(\frac{\epsilon E}{\mu^4} \right)^{2n-\sigma_n}, \quad n = 1, \dots \quad (7.56)$$

where

$$q^{(n)}(t) = \sum_{i=0}^n \bar{s}_i^{(n-i, n-i)}(t) \quad (7.57)$$

In the right-hand side of (7.57) contributions to all the first n -modes are taken into account, and in each of them approximations to the exact values of the amplitudes and the phase are taken with orders not exceeding $(n-i)$

Remark. C_n is a dimensionless constant, the factor $\frac{\sqrt{E}}{\mu}$ is introduced by the dimension analysis, $\frac{\epsilon E}{\mu^4}$ is a dimensionless factor, σ_n depends on n .

Proof follows from the estimates (5.22) and (7.32). Indeed, from (5.22) it follows that

$$|q(t) - q_n(t)| \leq \frac{8\sqrt{2E}}{k\mu} \left(\frac{2\epsilon E}{\mu^4} \right)^{2 - \frac{2n+1}{2}c}, \quad n = 1, \dots \quad (7.58)$$

According to (5.7)

$$q_n(t) = s_0(t) + s_1(t) + \dots + s_n(t), \quad (7.59)$$

and in (7.57) we take the approximation of $s_0^{(n)}(t)$ to $q_0(t)$ and the approximation $s_1^{(n-1, n-1)}$, for $s_1(t)$ and so on. We get

$$\begin{aligned} q(t) - q^{(n)}(t) &= q(t) - s_0(t) - \dots - s_n(t) \\ &+ s_0(t) - \bar{s}_0^{(n)}(t) + s_1(t) - \bar{s}_1^{(n-1, n-1)}(t) + \dots + s_n(t) - \bar{s}_n^{(0,0)}(t), \end{aligned} \quad (7.60)$$

and consequently

$$\begin{aligned} |q(t) - q^{(n)}(t)| &< |q(t) - q_n(t)| + |q_0(t) - \bar{s}_0^{(n)}(t)| \\ &+ |s_1(t) - \bar{s}_1^{(n-1, n-1)}(t)| + \dots + |s_n(t) - \bar{s}_n^{(0,0)}(t)| \end{aligned} \quad (7.61)$$

In order to finish the proof we need estimates (7.11) and the estimate (B.9) for (B.3). Having taken into account these relations we get (7.56).

8. Conclusion

The method for approximate solution to nonlinear dynamics equations in the rolling regime is presented. It is shown that in order to improve perturbation theory in the rolling regime it turns out to be effective not to use an expansion in trigonometric functions as it is done in case of small oscillations but use expansions in hyperbolic functions instead. In particular the Higgs equation in the rolling regime is considered. This regime is investigated using the representation of the solution in terms of elliptic functions. An accuracy of the corresponding approximation is estimated.

In this paper we have investigated the rolling regime for motions starting from the top of the Higgs potential, see Fig.1.A. The same method can be used for motion starting with zero velocity from the side that is near to the top, see Fig.1.C.

As to possible cosmological applications of rolling solutions it is worth to mention that in cosmology various rolling solutions are widely used. There is a notion of the slow roll regime that means that one can ignore the second order derivatives in the equation of motion [7, 8]. The rolling from the top of the potential is not necessary described by the slow roll approximation and methods similar to ones developed in this paper seems to be rather suitable [10].

There are rolling solutions also in the nonlocal cosmology [19, 20, 21, 22, 23]. These solutions have their analogue in the flat space-time [24, 25, 26]. Non slow roll regime is important in inflation and it is related with so-called stretch effect in nonlocal theories [27, 28, 29, 30].

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A. Proof of Lemma 1

Consider first in the case $n = 1$. The difference in the left-hand side of (4.22) is

$$\Delta_1 \equiv \text{cn}(\Omega t - \mathbf{K}, k) - \frac{\pi}{k\mathbf{K}'} \frac{\sinh\left(\frac{\pi\Omega t}{2\mathbf{K}'}\right)}{\cosh\left(\frac{\pi\mathbf{K}}{2\mathbf{K}'}\right)} \quad (\text{A.1})$$

Using (4.11) and the majorant of the following sum

$$r_1 \equiv \frac{\pi}{k\mathbf{K}'} \left\{ -\frac{\sinh(3\rho' u'')}{\cosh(3\rho')} + \frac{\sinh(5\rho' u'')}{\cosh(5\rho')} + \dots \right\} \quad (\text{A.2})$$

where

$$u'' = \frac{\Omega t}{\mathbf{K}}, \quad \rho' = \frac{\pi\mathbf{K}}{2\mathbf{K}'}, \quad u''\rho' = u' = \frac{\pi\Omega t}{2\mathbf{K}'}, \quad (\text{A.3})$$

we get

$$|r_1| \leq \frac{\pi}{k\mathbf{K}'} \sum_{j=1}^{\infty} \frac{\sinh((2j+1)\rho' u'')}{\cosh((2j+1)\rho')} < \frac{\pi}{k\mathbf{K}'} e^{-\rho'(1-u'')} \sum_{j=1}^{\infty} e^{-2j\rho'(1-u'')}.$$

Using

$$\frac{\pi}{k\mathbf{K}'} e^{-\rho'(1-u'')} \sum_{j=1}^{\infty} e^{-2j\rho'(1-u'')} = \frac{\pi}{k\mathbf{K}'} \frac{e^{-3\rho'(1-u'')}}{1 - e^{-2\rho'(1-u'')}}, \quad (\text{A.4})$$

we get

$$|r_1| \leq d_1, \quad (\text{A.5})$$

where

$$d_1 \equiv \frac{\pi}{k\mathbf{K}'} \frac{e^{-3\rho'(1-u'')}}{1 - e^{-2\rho'(1-u'')}}, \quad (\text{A.6})$$

In Fig.7. A. the bound (A.5) is shown.

We notice that the series in (A.4) converges if

$$e^{-\rho'(1-u'')} < 1, \quad (\text{A.7})$$

that is true if

$$0 < u'' < 1, \quad (\text{A.8})$$

since

$$\rho' = \frac{\pi \mathbf{K}}{2\mathbf{K}'} > 0 \quad (\text{A.9})$$

Since $u'' = \frac{\Omega t}{\mathbf{K}}$ the relation (A.8) means that

$$0 < \frac{\Omega t}{\mathbf{K}} < 1 \quad (\text{A.10})$$

Consider the right-hand side of (A.5),

$$d_1 \equiv \frac{\pi}{k\mathbf{K}'} \frac{e^{-3\rho'(1-u')}}{1 - e^{-2\rho'(1-u')}} = \frac{\pi}{k\mathbf{K}'} \frac{\exp\left(-3\frac{\pi\mathbf{K}}{2\mathbf{K}'}\right) \exp\left(\frac{3\pi\Omega t}{2\mathbf{K}'}\right)}{1 - \exp\left(-2\frac{\pi\mathbf{K}}{2\mathbf{K}'}\right) \exp\left(\frac{2\pi\Omega t}{2\mathbf{K}'}\right)} \quad (\text{A.11})$$

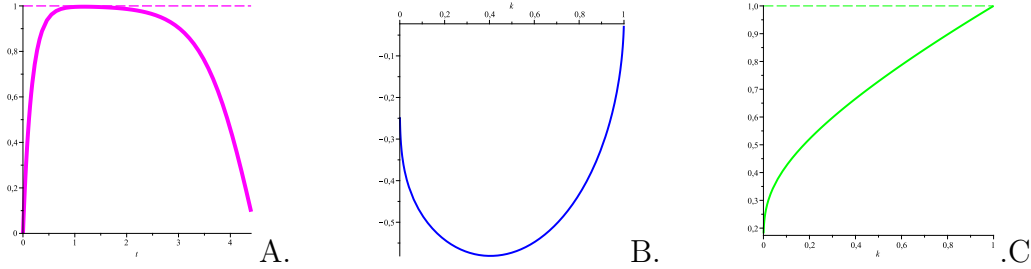


Figure 7: A. The estimate (A.5) is shown. Along the horizontal axis time values t are plotted (with $\Omega = 1$), and on vertical axis the relation $\frac{|r_1|}{d_1}$ is plotted. B. The estimate (A.12) is shown. Along the horizontal axis time values k are plotted, and on vertical axis $L(k)$ given by the relation (A.12) is plotted. C. The estimate (A.13) is shown. Along the horizontal axis time values k are plotted, and on vertical axis $\frac{\pi}{2\mathbf{K}'}$ is plotted.

Using explicit estimates (see. Fig. 7.B and Fig.7.C, respectively)

$$L(k) \equiv \exp\left(-\frac{\pi\mathbf{K}}{2\mathbf{K}'}\right) - \sqrt{1-k^2} \leq 0 \quad (\text{A.12})$$

$$\frac{\pi}{2\mathbf{K}'} \leq 1 \quad (\text{A.13})$$

we have

$$\exp\left(-\frac{\pi\mathbf{K}}{2\mathbf{K}'}\right) \exp\left(\frac{\pi\Omega t}{2\mathbf{K}'}\right) < k' e^{\Omega t} \quad (\text{A.14})$$

We require that

$$\left(k'\right)^2 \exp(2\Omega t) < \frac{1}{2} \quad (\text{A.15})$$

to estimate (A.11).

Then we have

$$d_1 \leq 2 \frac{\pi}{k\mathbf{K}'} \left(k'\right)^3 e^{3\Omega t} < \frac{4}{k} \left(k'\right)^3 e^{3\Omega t} \quad (\text{A.16})$$

Notice that for $k' < 1/2$

$$\Omega = \mu \frac{1}{\sqrt{1-2k'^2}} \leq \mu\sqrt{2} \quad (\text{A.17})$$

and (A.15) are true if

$$\left(k'\right)^2 \exp(2\Omega t) \leq \left(k'\right)^2 \exp\left(2\sqrt{2}\mu t\right) \leq \frac{1}{2}, \quad k' \exp\left(\sqrt{2}\mu t\right) \leq \frac{1}{\sqrt{2}} \quad (\text{A.18})$$

Consequently (A.18) is true, for

$$t < \frac{c}{\mu\sqrt{2}} \ln 1/k', \quad c < 1/2. \quad (\text{A.19})$$

Indeed,

$$k' \exp\left(\sqrt{2}\mu t\right) < (k')^{1-c} < 1/\sqrt{2} \quad \text{if } c < 1/2, \quad k' < 1/2. \quad (\text{A.20})$$

With (A.19) the following is true

$$d_1 \leq \frac{4}{k} \left(k'\right)^3 e^{3\Omega t} < \frac{4}{k} (k')^{3(1-c)}. \quad (\text{A.21})$$

In the same way for

$$r_n \equiv \frac{\pi}{k\mathbf{K}'} \left\{ -\frac{\sinh((2n+1)\rho'u'')}{\cosh((2n+1)\rho')} + \frac{\sinh((2n+3)\rho'u'')}{\cosh((2n+3)\rho')} + \dots \right\} \quad (\text{A.22})$$

we have

$$|r_n| \leq d_n, \quad (\text{A.23})$$

where

$$d_n \equiv \frac{\pi}{k\mathbf{K}'} \frac{e^{-(2n+1)\rho'(1-u'')}}{1 - e^{-2\rho'(1-u'')}} , \quad (\text{A.24})$$

and

$$d_n \leq \frac{4}{k} \left(k'\right)^{2n+1} e^{(2n+1)\Omega t} < \frac{4}{k} (k')^{(2n+1)(1-c)}. \quad (\text{A.25})$$

For small $k' < 1/2$ we get $k > 1/2$ and

$$d_1 \leq \frac{4}{k} \left(k'\right)^3 e^{3\Omega t} \leq \frac{8}{k} \left(k'\right)^3 e^{3\Omega t}. \quad (\text{A.26})$$

Finally for $k' < 1/2$ $k > 1/2$ we get

$$d_1 \leq \frac{4}{k} \left(k'\right)^3 e^{3\Omega t} \leq 8 \left(k'\right)^3 e^{3\Omega t}. \quad (\text{A.27})$$

B. Bounds of Δ

Having introduced notations

$$\Delta_n = \mu t \delta_n, \quad (\text{B.1})$$

we obtain:

$$\delta_0 = 1 - \frac{\pi}{2\mathbf{K}'} \frac{1}{\sqrt{1-2k'^2}} \quad (\text{B.2})$$

$$\delta_1 = \left(1 + \frac{3A^2}{8\mu^2}\epsilon\right) - \frac{\pi}{2\mathbf{K}'} \frac{1}{\sqrt{1-2k'^2}} \quad (\text{B.3})$$

$$\begin{aligned} &= 1 + \frac{3}{8} \left(1 + \frac{1}{1-2k'^2}\right) \left(\frac{\frac{\pi}{k\mathbf{K}'}}{\cosh\left(\frac{\pi\mathbf{K}}{2\mathbf{K}'}\right)}\right)^2 - \frac{\pi}{2\mathbf{K}'} \frac{1}{\sqrt{1-2k'^2}} \\ \delta_2 &= 1 + \frac{3A^2}{8\mu^2}\epsilon - \frac{15A^4}{2^8\mu^4}\epsilon^2 - \frac{\pi}{2\mathbf{K}'} \frac{1}{\sqrt{1-2k'^2}} \end{aligned} \quad (\text{B.4})$$

$$\begin{aligned} &= 1 + \frac{3}{8} \left(1 + \frac{1}{1-2k'^2}\right) \left(\frac{\frac{\pi}{k\mathbf{K}'}}{\cosh\left(\frac{\pi\mathbf{K}}{2\mathbf{K}'}\right)}\right)^2 - \frac{15}{2^8} \left(1 + \frac{1}{1-2k'^2}\right) \left(\frac{\frac{\pi}{k\mathbf{K}'}}{\cosh\left(\frac{\pi\mathbf{K}}{2\mathbf{K}'}\right)}\right)^4 \\ &\quad - \frac{\pi}{2\mathbf{K}'} \frac{1}{\sqrt{1-2k'^2}} \end{aligned}$$

These formulae are based on the relations:

$$\frac{\pi\Omega}{2\mathbf{K}'} = \frac{\pi}{2\mathbf{K}'} \frac{\mu}{\sqrt{1-2k'^2}} \quad (\text{B.5})$$

$$\frac{A^2}{\mu^2}\epsilon = \left(1 + \frac{1}{1-2k'^2}\right) \left(\frac{\frac{\pi}{k\mathbf{K}'}}{\cosh\left(\frac{\pi\mathbf{K}}{2\mathbf{K}'}\right)}\right)^2 \quad (\text{B.6})$$

We have (see Fig. 5)

$$\delta_0 < \sqrt{2}k'^2 \quad , 0 \leq k' \leq \frac{1}{2} \quad (\text{B.7})$$

and, consequently,

$$\Delta_0 < \sqrt{2}\mu k'^2 t \quad , 0 \leq k' \leq \frac{1}{2}. \quad (\text{B.8})$$

We also have

$$\delta_1 < k'^4. \quad (\text{B.9})$$

This estimation is illustrated in Fig.8.A.

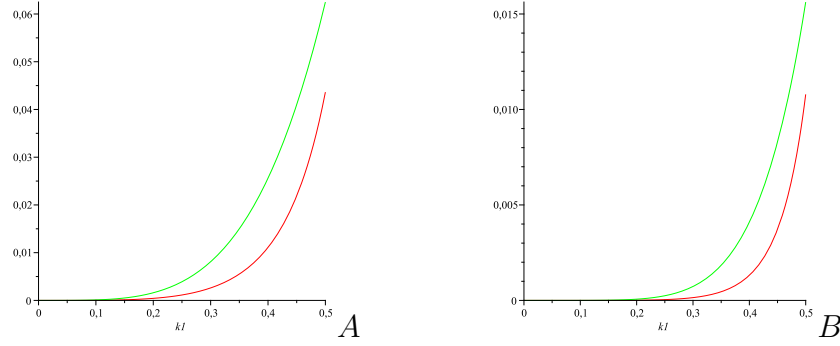


Figure 8: (color online) A. The plot illustrates the estimation (B.9). On the horizontal axis k'^2 is plotted, on the vertical axis the values of the left-hand side of (B.3) (red color) and the function k'^4 (green color) are presented. B. The plot illustrates the estimation (B.4). On the horizontal axis k'^2 is plotted, on the vertical axis the values of the left-hand side of (B.4) (red color) and the function k'^6 (green color) are presented.

From (B.9) we get that

$$\Delta_1 \leq \mu k'^4 t. \quad (\text{B.10})$$

In Fig.8.B. we show functions $\delta_2 = \delta_2(k')$ and k'^6 . From from this plot one can see that

$$\delta_2 < k'^6 \quad (\text{B.11})$$

and

$$\Delta_2 \leq \mu k'^6 t. \quad (\text{B.12})$$

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